Freie Universität Berlin
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# Crystalline Cohomology

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## Introduction

The purpose of this course is to provide an introduction to the basic theory of crystals and crystalline cohomology. Crystalline cohomology was invented by A.Grothendieck in 1966 to construct a Weil cohomology theory for a smooth proper variety $X$ over a field $k$ of characteristic $p > 0$. Crystals are certain sheaves on the crystalline site. The first main theorem which we are going to prove is that if there is a lift $X_W$ of $X$ to the Witt ring $W(k)$, then the category of integrable quasi-coherent crystals is equivalent to the category of quasi-nilpotent connection of $X_W/W$. Then we will prove that assuming the existence of the lift the crystalline cohomology of $X/k$ is "the same" as the de Rham cohomology of $X_W/W$. Following from this we will finally prove a base change theorem of the crystalline cohomology using the very powerful tool of cohomological descent. Along the way we will also see a crystalline version of a "Gauss-Manin" connection.

## References

[SP] Authors, *Stack Project*, `https://stacks.math.columbia.edu/download/crystalline.pdf`.

[BO] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Mathematical Notes **21**, Princeton University Press and University of Tokyo Press, 1978.

[B74] P. Berthelot, Cohomologie Cristalline des Schémas de Charactéristique $p > 0$, **LNM** 407, Springer Verlag, 1974.

# 1 INTRODUCTION (17/10/2017)

In this lecture we will give an introduction to crystals and crystalline cohomology. There will be no proofs, and the purpose is just to get a picture of what is going on.

# 2 DIVIDED POWERS (24/10/2017)

- The Definition of Divided Powers ([BO, §3, 3.1]).
- Examples: (a) If $A$ is an algebra over $\mathbb{Q}$; (b) If $A = W(k)$, the Witt ring of a perfect field $k$.
- Interlude: The Witt ring of a perfect field $k$ is characterized by the property that it is a complete DVR with uniformizer $p$ and residue field $k$.
- PD-ideals are nil ideals if $A$ is kill by $m \in \mathbb{N}^+$. Easy proof: For any $x \in I$, we have $x^n = n!\gamma_n(x) = 0$ for $n \geq m$.
- Definition of sub P.D. ideals ([BO, §3, 3.4]).
- Lemma: If $(A, I, \gamma)$ is a P.D. ring and $J \subseteq A$ is an ideal, then there is a PD-structure $\bar{\gamma}$ on $\bar{I} \colon = I(A/J)$ such that $(A, I, \gamma) \to (A/J, \bar{I}, \bar{\gamma})$ is a PD-map iff $J \bigcap I \subseteq I$ is a sub PD-ideal ([BO, §3, 3.5]).
- Theorem: If $(A, M)$ is a pair, where $A$ is a ring and $M$ is an $A$-module, then there is triple $(\Gamma_A(M), \Gamma_A^+(M), \tilde{\gamma})$ with an $A$-linear map $\varphi : M \to \Gamma_A^+(M)$ which satisfy the universal property that if $(B, J, \delta)$ is any PD-$A$-algebra and $\psi \colon M \to J$ is $A$-linear, then there is a unique PD-morphism

  $$\bar{\psi} \colon (\Gamma_A(M), \Gamma_A^+(M), \tilde{\gamma}) \to (B, J, \delta)$$

  such that $\bar{\psi} \circ \phi = \psi$. Moreover, we know that $\Gamma_A(M)$ is graded with $\Gamma_0 = A$ and $\Gamma_1 = M$.
- Sketch of the proof: We take $G_A(M)$ to be the $A$-polynomial ring generated by indeterminates $\{(x, n) | x \in M, n \in \mathbb{N}\}$ whose grading is given by $\deg(x, n) = n$. Let $I_A(M)$ be the ideal of $G_A(M)$ generated by elements

  1. $(x, 0) - 1$
  2. $(\lambda x, n) - \lambda^n (x, n)$ for $x \in M$ and $\lambda \in A$
  3. $(x, n)(x, m) - \frac{(n+m)!}{n!m!}(x, n+m)$
  4. $(x + y, n) - \sum_{i+j=n} (x, i)(y, j)$

  One sees that $I_A(M)$ is a homogeneous ideal. Define $\Gamma_A(M) := G_A(M)/I_A(M)$. Now let $x^[n]$ be the image of $(x, n)$. Then we have the following
- Lemma: The ideal $\Gamma_A^+(M) \subset \Gamma_A(M)$ has a unique PD-structure $\gamma$ such that $\gamma_i(x^{[1]}) = x^{[n]}$ for all $i \geq 1$ and all $x \in M$.

- Lemma: If $A'$ is an $A$-algebra, $A' \otimes_A \Gamma_A(M) \cong \Gamma_{A'}(A' \otimes_A M)$.
- Lemma: If $\{M_i | i \in I\}$ is a direct system of $A$-modules, then we have

$$\varinjlim_{i \in I} \Gamma_A(M_i) = \Gamma_A(\varinjlim_{i \in I} M_i)$$

- Lemma: $\Gamma_A(M) \otimes_A \Gamma_A(N) \cong \Gamma_A(M \oplus N)$.
- Lemma: Suppose $M$ is free with basis $S := \{x_i | i \in I\}$. Then $\Gamma_n(M)$ is free with basis $\{x_1^{[q_1]} \cdots x_k^{[q_k]} | \sum q_i = n\}$.

## 3 The PD-Envelop (07/11/2017)

**Theorem 3.1.** *Let $(A, I, \gamma)$ be a PD-algebra and let $J$ be an ideal in an $A$-algebra $B$ such that $IB \subseteq J$. Then there exists a $B$-algebra $\mathscr{D}_{B,\gamma}(J)$ with a PD-ideal $(\bar{J}, \bar{\gamma})$ such that $J\mathscr{D}_{B,\gamma}(J) \subseteq \bar{J}$, such that $\bar{\gamma}$ is compatible with $\gamma$, and with the following universal property: For any $B$-algebra containing an ideal $K$ which contains $JC$ and with a PD-structure $\delta$ compatible with $\gamma$, there is a unique PD-morphism $(\mathscr{D}_{B,\gamma}(J), \bar{J}, \bar{\gamma}) \to (C, K, \delta)$ making the obvious diagrams commute.*

*Proof.* First assume that $f(I) \subseteq J$. Viewing $J$ as a $B$-module we get a triple $(\Gamma_B(J), \Gamma_B^+(J), \tilde{\gamma})$. Let $\varphi\colon J \to \Gamma_1(J)$ be the canonical identification. We define a new ideal $\mathscr{J}$ generated by ideals of the two forms:

1. $\varphi(x) - x$ for $x \in J$
2. $\varphi(f(y))^{[n]} - f(\gamma_n(y))$ for $y \in I$.

One first has to show the following

**Lemma 3.2.** *The ideal $\mathscr{J} \bigcap \Gamma_B^+(J)$ is a sub PD-ideal of $\Gamma_B^+(J)$.*

So now we define $\mathscr{D}_{B,\gamma}(J)$ to be $\Gamma_B(J)/\mathscr{J}$, $\bar{J} := \Gamma_B^+(J)/\mathscr{J} \bigcap \Gamma_B^+(J)$, and $\tilde{\gamma}$ is the PD-structure induced by the sub PD-ideal. Now one checks the two things: $J\mathscr{D} \subseteq \bar{J}$ (come from (1) of the definition of $\mathscr{J}$), and $\gamma$ is compatible with $\gamma$ (follows from (2) of the definition of $\mathscr{J}$). Now it is easy to check that the triple $(\mathscr{D}_{B,\gamma}(J), \bar{J}, \mathscr{J})$ is universal among all such triples. □

Here is a list of important properties of PD-envelops.

- $\bar{J}$ is generated, as a PD-ideal, by $J$. That is $\bar{J}$ is generated by elements $\{\bar{\gamma}_n(j) | j \in J, n \geq 1\}$. Moreover a set of generators of $J$ provides a set of PD-generators of $\bar{J}$.
- If the map $(A, I, \gamma) \to (B, J)$ factors as a diagram

$$\begin{array}{ccc} (A, I, \gamma) & \longrightarrow & (B, J) \\ & \searrow \quad \nearrow & \\ & (A', IA', \gamma') & \end{array}$$

  then we have $\mathscr{D}_{B,\gamma}(J) = \mathscr{D}_{B,\gamma'}(J)$.

- The canonical map $B/J \to \mathscr{D}_{B,\gamma}(J)$ is an isomorphism. Indeed, one just has to consider the PD-triple $(B/J, 0, 0)$ and play with the universal property of $(D_{B,\gamma}(J), \bar{J}, \bar{\gamma})$.
- If $M$ is an $A$-module, if $B = \mathrm{Sym}_A(M)$, and if $\bar{J}$ is the ideal $\mathrm{Sym}_A^+(M)$, then $D_{B,\gamma}(J) = \Gamma_A(M)$. This is clear when man plays with the universal property of the PD-envelop of $(B, J)$.
- Lamma: Suppose that $J \subseteq B$ is an ideal, and $(A, I, \gamma) \to (B, J)$ is a morphism. If $B'$ is flat over $B$, then there is a canonical isomorphism $(\mathscr{D}_{B,\gamma} \otimes_B B') \xrightarrow{\cong} \mathscr{D}_{B',\gamma}(JB')$.
- Theorem: Let $(A, I, \gamma)$ be a PD-triple. Then there exists a unique PD-structure $\delta$ on the ideal $J = IA\langle x_t \rangle_{t\in T} + (A\langle x_t \rangle_{t\in T})_+$ such that
  1. $\delta_n(x_i) = x_i^{[n]}$;
  2. The map $(A, I, \gamma) \to (A\langle x_t \rangle_{t\in T}, J, \delta)$ is a PD-morphism.

  Moreover, there is a universal property: Whenever $(A, I, \gamma) \to (C, K, \epsilon)$ is a PD-map and $\{k_t\}_{t\in T}$ is a family in $K$, then there exists a unique PD-map $(A\langle x_t \rangle_{t\in T}, J, \delta) \to (C, K, \epsilon)$ sending $x_t \mapsto k_t$.
- Let $(B, I, \gamma)$ be a PD-triple, and let $J \subseteq B$ be an ideal containing $I$. Choose $\{f_t\}_{t\in T}$ a family in $J$ such that $J = I + \langle f_t \rangle_{t\in T}$. Then there exists a surjection $\psi : (B\langle x_t \rangle, J', \delta) \to (D_{B,\gamma}(J), \bar{J}, \bar{\gamma})$ which maps $x_t \mapsto \bar{f}_t$, where $(B\langle x_t \rangle, J', \delta)$ is the triple defined in the above theorem, and $\bar{f}_t$ is the image of $f_t$. The kernel of $\psi$ is generated by all elements:
  1. $x_t - f_t$ for $f_t \in J$;
  2. $\delta_n(\sum_t r_t x_t - r_0)$ whenever $\sum_t r_t f_t = r_0$ with $r_0 \in I$, $r_t \in B$ and $n \geq 1$.
- Lemma: Let $(A, I, \gamma)$ be a PD-ring. Let $B$ be an $A$-algebra, and let $IB \subseteq J \subseteq B$ be an ideal. Then we have

$$(D_{B[x_t],\gamma}(JB[x_t] + \langle x_t \rangle), \overline{JB[x_t] + \langle x_t \rangle}, \bar{\gamma}) = (D_{B,\gamma}(J)\langle x_t \rangle, J', \delta)$$

# 4 The Affine Crystalline Site (14/11/2017)

**Settings:** Let $p$ be a prime number, and let $(A, I, \gamma)$ be a PD-triple in which $A$ is a $\mathbb{Z}_{(p)}$-algebra (i.e. any integer which is prime to $p$ is invertible in in $A$). Let $A \to C$ be a ring map such that $IC = 0$ and $p$ is nilpotent in $C$. (Note that in this case $C$ is automatically an $A/I$-algebra.)
**Typical Examples:** Keep in mind the situation when

$$(A, I, \gamma) = (W(k), (p), \gamma)$$

and when

$$(A, I, \gamma) = (W_n(k), (p), \gamma)$$

where $k$ is a perfect field of characteristic $p > 0$.

**Definition 1.** 1. A thickening of $C$ over $(A, I, \gamma)$ is a PD-map $(A, I, \gamma) \to (B, J, \delta)$ such that $p$ is nilpotent in $B$, and an $A/I$-algebra map $C \to B/J$.

2. A map of PD-thickenings is a map $(B,J,\delta) \to (B',J',\delta')$ over the thickening $(A,I,\gamma)$ whose induced map $B/J \to B'/J'$ is a $C$-algebra map.

3. We denote $\mathrm{CRIS}(C/A)$ the category of PD-thickenings of $C$ over $(A,I,\gamma)$.

4. We denote $\mathrm{Cris}(C/A)$ the full subcategory of $\mathrm{CRIS}(C/A)$ whose objects are PD-thickenings $((B,J,\delta), C \to B/J)$ in which $C \to B/J$ is an isomorphism.

**Lemma 4.1.** *1. The category* $\mathrm{CRIS}(C/A)$ *has non-empty products, and the category* $\mathrm{Cris}(C/A)$ *has empty product, i.e. the terminal object.*

*2. The category* $\mathrm{CRIS}(C/A)$ *has all finite non-empty colimits and the functor*

$$\mathrm{CRIS}(C/A) \longrightarrow C-\text{algebras}$$

$$(B,J,\delta) \longrightarrow B/J$$

*commutes with those.*

*3. The category* $\mathrm{Cris}(C/A)$ *has all finite non-empty colimits and the functor*

$$\mathrm{Cris}(C/A) \longrightarrow \mathrm{CRIS}(C/A)$$

*commutes with those.*

*Proof.* (i) The empty product of $\mathrm{Cris}(C/A)$ is indeed $(C,0,\emptyset)$. The product of a family of thickenings $(B_t, J_t, \delta_t)$ in $\mathrm{CRIS}(C/A)$ is just $(\prod_t B_t, \prod_t J_t, \prod_t \delta_t)$ with the $A/I$-algebra map $C \to \prod_t B_t$ coming from each $C \to B_t$.

(ii) First note that by `https:https://stacks.math.columbia.edu/tag/04AS` to show colimits (resp. limit) exist we only have to prove that coproducts and pushouts (resp. products and pullbacks) exist. We divide the proof into steps.

- The category of PD-triples admits limits.
- The category of PD-triples admits colimits.
- Coproducts of pairs exist in $\mathrm{CRIS}(C/A)$. There are also two remarks: (a) If the pair is in $\mathrm{Cris}(C/A)$, then the coproduct is also in $\mathrm{Cris}(C/A)$. (b) The functor

$$\mathrm{CRIS}(C/A) \longrightarrow C-\text{algebras}$$

  commutes with coproducts.
- Coequalizers of pairs exist in $\mathrm{CRIS}(C/A)$. There are also two remarks: (a) If the pair is in $\mathrm{Cris}(C/A)$, then the coequalizer is also in $\mathrm{Cris}(C/A)$. (b) The functor

$$\mathrm{CRIS}(C/A) \longrightarrow C-\text{algebras}$$

  commutes with coproducts.
- Conclude the proof.

□

**Definition 2.** Let $\widehat{\text{Cris}}(C/A)$ be the category whose objects are PD-triples $(B, J, \delta)$, where $B$ is only $p$-adically comlete instead of nilpotent in $B$, plus an $A/I$-algebra map $C \to B/J$ as usual. Clearly that $\text{Cris}(C/A)$ is a full subcategory of $\widehat{\text{Cris}}(C/A)$, as $p^n$-torsion rings are $p$-adically complete.

**Lemma 4.2.** *Let $(A, I, \gamma)$ be a PD-ring. Let $p$ be a prime number. If $p$ is nilpotent in $A/I$, and if $A$ is a $\mathbb{Z}_{(p)}$-algebra then*

1. *The $p$-adic completion $\hat{A}$ goes surjectively to $A/I$.*
2. *The kernel of $\hat{A} \to A/I$ is $\hat{I}$.*
3. *Each $\gamma_n$ is continuous for the $p$-adic topology on $I$.*
4. *For $e$ large, the idea $p^e A \subseteq I$ is preserved by $\gamma_n$ and we have*

$$(\hat{A}, \hat{I}, \hat{\delta}) = \varprojlim_e (A/p^e A, I/p^e I, \gamma_e)$$

**Lemma 4.3.** *Let $P \to C$ be a surjection of $A$-algebras with kernel $J$. We write $(D, \bar{J}, \bar{\gamma})$ for the PD-envelop of $(P, J)$ with respect to $(A, I, \gamma)$. Let $(\hat{D}, \hat{\bar{J}}, \hat{\bar{\gamma}})$ be the completion of $(D, \bar{J}, \bar{\gamma})$. For every $e \geq 1$, set $(P_e, J_e) := (P/p^e P, J/(J \cap p^e P))$ and $(D_e, \bar{J}_e, \bar{\gamma}_e)$ the PD-envelop of this pair. Then for large $e$ we have*

1. *$p^e D \subseteq \bar{J}$ and $p^e \hat{D} \subseteq \hat{\bar{J}}$ are preserved by the PD-structures.*
2. *$\hat{D}/p^e \hat{D} \cong D/p^e D = D_e$ as PD-rings.*
3. *$(D_e, \bar{J}_e, \bar{\gamma}_e) \in \text{Cris}(C/A)$.*
4. *$(\hat{D}, \hat{\bar{J}}, \hat{\bar{\gamma}}) = \varprojlim (D_e, \bar{J}_e, \bar{\gamma}_e)$.*
5. *$(\hat{D}, \hat{\bar{J}}, \hat{\bar{\gamma}}) \in \hat{\text{Cris}}(C/A)$.*

**Lemma 4.4.** *Let $P$ be a polynomial algebra over $A$, and let $P \twoheadrightarrow C$ be a surjection of $A$-algebras with kernel $J$. Then every object $(B, J, \delta)$ of $\text{CRIS}(C/A)$ there exists an $e$ and a morphism*

$$(D_e, \bar{J}_e, \bar{\gamma}_e) \to (B, J, \delta)$$

*in* $\text{CRIS}(C/A)$.

**Lemma 4.5.** *Let $P$ be a polynomial algebra over $A$, and let $P \twoheadrightarrow C$ be a surjection of $A$-algebras with kernel $J$. Let $(D, \bar{J}, \bar{\gamma})$ be the $p$-adic completion of $D_{P,\gamma}(J)$. For every object $(B, J, \delta)$ of $\widehat{\text{Cris}}(C/A)$ there exists a morphism*

$$(D, \bar{J}, \bar{\gamma}) \to (B, J, \delta)$$

*in* $\widehat{\text{Cris}}(C/A)$.

# 5 The Differentials of PD-Structures (21/11/2017)

- Let $A$ be a ring. Let $(B,J,\delta)$ be a PD-triple. Let $A \to B$ be a ring map. Let $M$ be a $B$-module. A PD-derivation is a usual $A$-derivation $\theta\colon B \to M$ with the extra condition that

$$(*) \qquad \theta(\gamma_n(x)) = \gamma_{n-1}(x)\theta(x)$$

for all $n \geq 1$ and $x \in J$. Let

$$\Omega_{B/A,\delta} : \Omega_{B/A}/\langle d(\gamma_n(x)) - \gamma_{n-1}(x)dx\rangle$$

Then $\Omega_{B/A,\delta}$ has the universal property that

$$\mathrm{Hom}_B(\Omega_{B/A,\delta}, M) \xrightarrow{\cong} \text{PD-Der}_A(B,M)$$

where $M$ is a $B$-module. Conceptually, condition $(*)$ can be thought of as the following:

$$d(\frac{x^n}{n!}) = \frac{x^{n-1}}{(n-1)!}dx$$

- A basic Lemma:

**Lemma 5.1.** *Let $A$ be a ring. Let $(B,J,\delta)$ be a PD-triple, and $A \to B$ be a ring map.*

1. *If we equip $B[X]$ with the PD-structure $(B[X], JB[X], \delta')$, where*

$$\gamma_n(aX^m) = \gamma_n(a)X^{mn}$$

*then we have*

$$\Omega_{B[X]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[X] \oplus B[X]dX$$

*Here $B[X]dX$ just means a free $B[X]$-module.*

2. *If $B\langle X\rangle$ is equipped with the PD-structure $(JB\langle X\rangle + B\langle X\rangle_+, \delta')$, where $\delta'$ takes $j \in J$ to $\delta_n(j)$ and $jX^{[m]}$ to $\frac{(m+n)!}{m!n!} j^n X^{[m+n]}$, then*

$$\Omega_{B\langle X\rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle X\rangle \oplus B\langle X\rangle dX$$

3. *Let $K \subseteq J$ be an ideal preserved by $\delta_n$ for all $n \geq 1$. Set $B' := B/K$ and denote $\delta'$ the induced PD-structure on $J/K$. Then we an exact sequence:*

$$K/K^2 \to \Omega_{B/A,\delta} \otimes_B B' \to \Omega_{B'/A,\delta'} \to 0$$

*Proof.* (1) Set $B[X] \xrightarrow{d} \Omega_{B/A,\delta} \otimes_B B[X] \oplus B[X]dX$ sending

$$b_0 + b_1X + \cdots b_nX^n \mapsto db_0 \otimes 1 + db_1 \otimes X + \cdots + db_n \otimes X^n + b_1dX + \cdots + nb_nX^{n-1}dX$$

This is an $A$-derivation. For example we have the derivation:

$$\begin{aligned} d(\delta'_n(bX^m)) &= d(X^{mn}\delta_n(b)) \\ &= \delta_n(b)dX^{mn}dX + \delta_{n-1}(b)\cdot db\cdot X^{mn} \\ &= mX^{mn-1}\cdot(n\delta_n(b))dX + \delta_{n-1}(b)\cdot db\cdot X^{mn} \\ &= mX^{mn-1}\cdot\delta_{n-1}(b)\cdot b\cdot dX + \delta_{n-1}(b)\cdot db\cdot X^{mn} \\ &= (\delta_{n-1}(b)\cdot X^{m(n-1)})\cdot(X^m db + mbX^{m-1}dX) \\ &= (\delta'_{n-1}(bX^m)\cdot d(bX^m)) \end{aligned}$$

The universal property: Using the universal property of direct sum the universal property of $d$ boils down to the universal property of $\Omega_{B/A,\delta}$ and the universal property of the free module $B[X]dX$.

(2) Almost the same as (1).

(3) Look at the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \Omega_{B/A}\otimes_B B' & \longrightarrow & \Omega_{B'/A} & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow\!\!\!\downarrow \phi & & \downarrow\!\!\!\downarrow \varphi & & \\ 0 & \longrightarrow & M' & \longrightarrow & \Omega_{B/A,\delta}\otimes_B B' & \longrightarrow & \Omega_{B'/A,\delta'} & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Ker}(\phi) \twoheadrightarrow \mathrm{Ker}(\varphi)$, we see that $f$ is surjective. Since $K/K^2 \twoheadrightarrow M$, it follows that $K/K^2 \twoheadrightarrow M'$. □

- Definition: Let $(A, I, \gamma)$ be a PD-ring. We denote $I^{[n]}$ the ideal generated by $\gamma_{e_1}(x_1)\cdots\gamma_{e_t}(x_t)$ with $\sum e_t \geq n$ and $x_i \in I$. So we have $I^{[0]} = A$, $I^{[1]} = I$ and $I^i \subseteq I^{[i]}$.

- Here is an important Proposition:

**Proposition 5.2.** *Let $a\colon (A, I,\gamma) \to (B, J,\delta)$ be a map of PD-triples. Let $(B(1), J(1),\delta(1))$ be the coproduct of $a$ with itself. Denote $K$ the kernel of the diagonal map $\Delta\colon B(1) \to B$. Then we have*

$$\Omega_{B/A,\delta} \cong K/(K^2 + (K\bigcap J(1))^{[2]})$$

*Proof.* Let's denote the two projections

$$B \underset{s_1}{\overset{s_0}{\rightrightarrows}} B(1)$$

by $s_0, s_1$ respectively. Since the composition

$$B \underset{s_1}{\overset{s_0}{\rightrightarrows}} B(1) \xrightarrow{\Delta} B$$

is the identity, we see that the map $B \to B(1)$ sending $b \mapsto s_0(b) - s_1(b)$ factors through K. Thus we obtain a map

$$d\colon B \longrightarrow K/(K^2 + (K\cap J(1))^{[2]})$$

Clearly $d$ is additive and vanishes on $A$, and

$$\begin{aligned}
d(b_1 b_2) &= b_1 d(b_2) + b_2 d(b_1) \\
&= s_1(b_1)(s_1(b_1) - s_0(b_2)) + s_0(b_2)(s_1(b_1) - s_0(b_1)) \\
&= s_1(b_1)s_1(b_2) - s_0(b_2)s_0(b_1) \\
&= s_1(b_1 b_2) - s_0(b_1 b_2)
\end{aligned}$$

Thus $d$ is a derivation. We have to check that $d$ is a PD-derivation. Let $x \in J$. Set $y = s_1(x)$, $z = s_0(z)$ and $\lambda := \delta(1)$. Since $d(\lambda_n(x)) = s_1(\lambda_n(x)) - s_0(\lambda_n(x)) = \lambda_n(y) - \lambda_n(z)$, and $\lambda_{n-1}(x) \cdot dx = \lambda_{n-1}(y)(y-z)$, we need to show that

$$\lambda_n(y) - \lambda_n(z) = \lambda_{n-1}(y)(y-z)$$

for all $n \geq 1$. If $n = 1$ this is clearly true. Let $n > 1$. We have that

$$\lambda_n(z-y) = \sum_{i=0}^{n} (-1)^{n-i} \lambda_i(z) \lambda_{n-i}(y) \in K^2 + (K \cap J(1))^{[2]}$$

as $z - y \in K \cap J(1)$ and $n \geq 2$. Then we have

$$\begin{aligned}
\lambda_n(y) - \lambda_n(z) &= \lambda_n(y) + \sum_{i=0}^{n-1} (-1)^{n-i} \lambda_i(z) \lambda_{n-i}(y) \\
&= \lambda_n(y) + (-1)^n \lambda_n(y) + \sum_{i=1}^{n-1} (-1)^{n-i} (\lambda_i(y) - \lambda_{i-1}(y)(y-z)) \lambda_{n-i}(y)
\end{aligned}$$

Since we have

$$\lambda_i(y)\lambda_{n-i}(y) = \binom{n}{i} \lambda_n(y)$$

and

$$\lambda_{i-1}(y)\lambda_{n-i}(y) = \binom{n-1}{i-1} \lambda_{n-1}(y)$$

we can continue

$$\begin{aligned}
\lambda_n(y) - \lambda_n(z) &= \lambda_n(y) + (-1)^n \lambda_n(y) + \sum_{i=1}^{n-1} (-1)^{n-i} \binom{n}{i} \lambda_n(y) - \sum_{i=1}^{n-1} (-1)^{n-i} \binom{n-1}{i-1} \lambda_{i-1}(y)(y-z)) \\
&= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \lambda_n(y) - \sum_{i=0}^{n-2} (-1)^{n-i-1} \binom{n-1}{i} \lambda_{n-1}(y)(y-z) \\
&= (1-1)^n \lambda_n(y) - (1-1)\lambda_{n-1}(y-z) + \lambda_{n-1}(y)(y-z) \\
&= \lambda_{n-1}(y)(y-z)
\end{aligned}$$

Let $M$ be any $B$-module, and let $\theta \colon B \to M$ be a PD $A$-derivation. Set $D := B \oplus M$, where $M$ is an ideal of square 0. Define a PD-structure on $J \oplus M \subseteq D$ by setting $\delta'_n(x+m) = \delta'_n(x) + \delta'_{n-1}(x)m$ for all $n \geq 1$. There are two PD-morphisms:

$$(B, J, \delta) \underset{t_1}{\overset{t_0}{\rightrightarrows}} (D = B \oplus M, J \oplus M, \delta')$$

where $t_1$ is just the canonical inclusion $b \mapsto b$ and $t_2$ is the map sending $b \mapsto b + \theta(b)$. Thus by the universal property we have a commutative diagram

$$\begin{array}{ccc} (B(1), J(1), \delta(1)) & \longrightarrow & (D, J \oplus M, \delta') \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ (B = B(1)/K, J, \delta) & = & (B, J, \delta) \end{array}$$

This induces a map $K \to M$. Since $M^2 = 0$ and $M^{[2]} = 0$. Thus we get a factorization

$$\phi := K/(K + (K \cap J(1))^{[2]}) \to M$$

This $\phi$ is compatible with $d$ and $\theta$ by construction, and it is unique because $K$ is generated by $\{s_1(b) - s_0(b) | b \in B\}$. □

- Lemma: Let $(B, J, \delta) \in \mathrm{CRIS}(C/A)$ and let $(B(1), J(1), \delta(1))$ be the coproduct in $\mathrm{CRIS}(C/A)$. Let $K$ be the kernel of the diagonal. Then $K \cap J(1) \subseteq J(1)$ is preserved by the PD-structure and,
$$\Omega_{B/A,\delta} \cong K/(K^2 + (K \bigcap J(1))^{[2]})$$

## 6 The de Rham Complex in the Affine Case (28/11/2017)

- Lemma: Let $(A, I, \gamma)$ be a PD-triple, and let $A \to B$ be a ring map. Let $IB \subseteq J \subseteq B$ be an ideal. Let $(D, \bar{J}, \bar{\gamma}) := (D_{B,\gamma}(J), \bar{J}, \bar{\gamma})$. Then we have
$$\Omega_{D/A,\delta} = \Omega_{B/A} \otimes_B D$$

- *Proof.* Let's first suppose that $A \to B$ is flat. Then there is a unique PD-structure $(B, IB, \gamma')$ which is compatible with $(A, I, \gamma)$. By a lemma in §3, we see that there is a surjective morphism
$$(B\langle x_t \rangle, J', \gamma') \longrightarrow (D, \bar{J}, \bar{\gamma})$$
where $J' := JB\langle x_t \rangle + B\langle x_t \rangle_+$, whose kernel is generated by elements of the forms: $(x_t - f_t)$, and $\gamma'_n(\sum_t r_t f_t - r_0)$ where $r_t \in B$ and $r_0 \in IB$. Since we have that
$$\Omega_{B\langle x_t \rangle/A} \cong \Omega_{B/A} \otimes_B B\langle x_t \rangle \oplus B\langle x_t \rangle dx_t$$
Thus we have
$$\Omega_{B\langle x_t \rangle/A} \otimes_{B\langle x_t \rangle} D \cong \Omega_{B/A} \otimes_B D \oplus D dx_t$$
By 5.1 there is a canonical surjection
$$\Omega_{B\langle x_t \rangle/A} \otimes_{B\langle x_t \rangle} D \longrightarrow \Omega_{D/A}$$
whose kernel is generated by all $\{dk \otimes 1 | k \in \mathrm{Ker}(B\langle x_t \rangle \twoheadrightarrow D)\}$.

Clearly the canonical composition:

$$\Omega_{B/A}\otimes_B D \hookrightarrow \Omega_{B\langle x_t\rangle/A}\otimes_{B\langle x_t\rangle} D \twoheadrightarrow \Omega_{B\langle x_t\rangle/A}\otimes_{B\langle x_t\rangle} D/(d(x_t-f_t))_{t\in T}$$

is surjective. But since it has a retraction, it is an isomorphism. Now to prove the lemma we only need to show that

$$\lambda := \Omega_{B\langle x_t\rangle/A}\otimes_{B\langle x_t\rangle} D/(d(x_t-f_t)_{t\in T}) \twoheadrightarrow \Omega_{D/A,\delta}$$

is an isomorphism. Given an element $\gamma'_n(\sum_{t\in T} r_t x_t - r_0)$ satisfying the relation $\sum_{t\in T} r_t f_t - r_0$ with $r_t\in B$ and $r_0\in IB$, we have

$$\begin{aligned} d\gamma'_n(\sum_{t\in T} r_t x_t - r_0) &= \gamma'_{n-1}(\sum_{t\in T} r_t x_t - r_0)d(\sum_{t\in T} r_t x_t - r_0)\\ &= \gamma'_{n-1}(\sum_{t\in T} r_t x_t - r_0)(\sum_{t\in T} r_t d(x_t-f_t) - \sum_{t\in T}(x_t-f_t)dr_t) \end{aligned}$$

is 0 in $\Omega_{B\langle x_t\rangle/A}\otimes_{B\langle x_t\rangle} D/(d(x_t-f_t)_{t\in T})$. But since those elements generate the kernel of $\lambda$, we conclude that $\lambda$ is an isomorphism.

In the general case we write $B$ as a quotient $P\twoheadrightarrow B$ of a polynomial $P$ over $A$. Let $J'\subseteq P$ be the inverse image of $J$, and let $(D',\bar{J}',\delta)$ be the PD-envelop of $(P,J')$. Then there is a surjection

$$(D',\bar{J}',\delta)\twoheadrightarrow(D,\bar{J},\bar{\gamma})$$

whose kernel is generated by $\{\delta_n(k)|k\in K := \mathrm{Ker}(P\twoheadrightarrow B)\}$. But since $P$ is flat over $A$ we have

$$\Omega_{D'/A,\delta} = \Omega_{P/A}\otimes_P D'$$

The kernel $M$ of

$$\Omega_{P/A}\otimes_P D = \Omega_{D'/A,\delta}\otimes_{D'} D \to \Omega_{D/A,\bar{\gamma}'}$$

is generated by $\{d\delta_n(k)\otimes 1|k\in K\}$. Since $d\delta_n(k) = \delta_{n-1}(k)dk$, the kernel $M$ is actually generated by $\{dk\otimes 1|k\in K\}$. As $\Omega_{B/A}$ is the quotient of $\Omega_{P/A}\otimes_P B$ by the submodule generated by $\{dk\otimes 1|k\in K\}$, we have that $\Omega_{B/A}\otimes_B D\twoheadrightarrow\Omega_{D/A,\bar{\gamma}'}$ is an isomorphism. □

- Let $B$ be a ring, and let $\Omega_B := \Omega_{B/\mathbb{Z}}$. Let $d\colon B\to\Omega_B$ be the canonical derivation. Set $\Omega^i_B := \bigwedge^i_B \Omega_B$. The we get a complex

$$0\to\Omega^0_B \xrightarrow{d^0} \Omega^1_B \xrightarrow{d^1} \Omega^2_B \xrightarrow{d^2} \cdots$$

where the differentials $d^p\colon \Omega^p_B \longrightarrow \Omega^{p+1}_B$ is defined by

$$d(b_0 db_1 \bigwedge db_2 \bigwedge\cdots\bigwedge db_p) \longrightarrow db_0 \bigwedge db_1 \bigwedge db_2 \bigwedge\cdots\bigwedge db_p)$$

Clearly we have that $d\circ d = 0$, so this is a complex if we can show that $d$ is well-defined.

Indeed, the $B$-module $\Omega_{B/\mathbb{Z}}$ is the free module on the basis $\{db|b\in B\}$ modulo the sub $B$-module $M$ generated by elements of the form $d(a+b)-da-db$ and $d(ab)-adb-bda$. If we regard $M$ as a sub abelian group of the free $B$-module, then $M$ is generated by $sd(a+b)-sda-sdb$ and $sd(ab)-sadb-sbda$ with $s\in B$. These are mapped to 0 by the map we defined. So $d^1$ is well-defined. The map $d^1$ defines for us a map

$$\psi\colon \underbrace{\Omega_B\otimes_{\mathbb{Z}}\Omega_B\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}\Omega_B}_{p-\text{times}}\longrightarrow\Omega^{p+1}$$

sending

$$w_1\otimes\cdots\otimes w_p\mapsto\sum_i(-1)^{(i+1)}w_1\wedge\cdots\wedge dw_i\wedge\cdots\wedge w_p$$

To show that $d^p$ is well-defined we only have to show that $\psi$ sends

$$w_1\otimes\cdots\otimes fw_i\otimes\cdots\otimes w_p-w_1\otimes\cdots\otimes fw_j\otimes\cdots\otimes w_p$$

to 0 for all $f\in B$. The following equations

$$\begin{aligned}&d(fa_1)\wedge db_1\wedge a_2db_2-fa_1db_1\wedge da_2\wedge db_2-da_1\wedge db_1\wedge fa_2db_2+a_1db_1\wedge dfa_2\wedge db_2\\=&(a_2dfa_1+fa_1da_2-fa_2da_1-a_1dfa_2)\wedge db_1\wedge db_2\\=&0\end{aligned}$$

shows without the loss of generality that $w_1\otimes\cdots\otimes fw_i\otimes\cdots\otimes w_p-w_1\otimes\cdots\otimes fw_j\otimes\cdots\otimes w_p$ is mapped to 0. So we win.

- Lemma: Let $B$ be a ring. Let $\pi:\Omega_B\to\Omega$ be a surjection of $B$-modules. Denote $d:B\to\Omega$ be the composition of the derivation $d_B:=B\to\Omega_B$ with the surjection. Set $\Omega^i=\bigwedge^i_B(\Omega)$. Assume that $\mathrm{Ker}(\pi)$ is generated as a $B$-module by some elements $\omega\in\Omega_B$ such that $d^1_B(\omega)$ is in the kernel of $\Omega^2_B\twoheadrightarrow\Omega^2$. Then there is a (de Rham) complex

$$\Omega^0\to\Omega^1\to\cdots$$

whose differentials are defined by

$$d^p:\Omega^p\to\Omega^{p+1},\qquad d^p(fw_1\wedge\cdots\wedge w_p)\mapsto d^p(f)\wedge w_1\wedge\cdots\wedge w_p$$

- *Proof.* We only have to prove that there exist commutative diagrams:

$$\begin{array}{ccccccc}B & \xrightarrow{d_B} & \Omega_B & \xrightarrow{d^1_B} & \Omega^2_B & \xrightarrow{d^2_B} & \cdots\\ \| & & \downarrow\pi & & \downarrow\wedge^2\pi & & \\ B & \xrightarrow{d} & \Omega & \xrightarrow{d^1} & \Omega^2 & \xrightarrow{d^2} & \cdots\end{array}$$

The left square is given by definition. For the second square we have to show that $\mathrm{Ker}(\pi)$ goes to $\mathrm{Ker}(\wedge^2\pi)$ under $d^1_B$. But $\mathrm{Ker}(\pi)$ is generated by $bw$, where $b\in B$ and $d^1_Bw\in\mathrm{Ker}(\wedge^2\pi)$, and $d^1_B(bw)=d_Bb\wedge w+bd^1_Bw\in\mathrm{Ker}(\wedge^2\pi)$ as desired.

If $i>1$, then we have that $\mathrm{Ker}(\wedge^i\pi)$ is equal to the image of

$$\mathrm{Ker}(\pi)\otimes\Omega^{(i-1)}\to\Omega^i$$

Now let $w_1 \in \text{Ker}(\pi)$ and $w_2 \in \Omega_B^{(i-1)}$. We have

$$d_B^i(w_1 \wedge w_2) = d_B^1 w_1 \wedge w_2 - w_1 \wedge d_B^{(i-1)} w_2$$

which is seen by the induction hypothesis to be contained in $\text{Ker}(\wedge^{(i+1)}\pi)$. □

- Now we consider a special case when $\Omega := \Omega_{B/A,\delta}$, where $B$ is an $A$-algebra equipped with a PD-structure $(B, J, \delta)$. In this case the kernel of $\Omega_{B/\mathbb{Z}} \to \Omega_{B/A,\delta}$ is generated by elements of the form $d_B a$ for $a \in A$ and $d_B\delta_n(x) - \delta_{n-1}(x)d_B x$ for $x \in J$. It is enough to show that the image of these elements under $d_B^1$ is contained in $\text{Ker}(\wedge^2\pi)$. But we have

$$d_B^1(d_B a) = 0, \ \ \forall a \in A$$

and,

$$\begin{aligned} d_B^1(d_B\delta_n(x) - \delta_{n-1}(x)d_B x) &= -d_B^1(\delta_{n-1}(x)d_B x)) \\ &= -d_B(\delta_{n-1}(x)) \wedge d_B(x) \\ &= -\delta_{n-2}(x)d_B x \wedge d_B x \\ &= 0 \end{aligned}$$

This proves everything.

- Integrable connections and the induced de Rham Complex.

# 7 THE CRYSTALLINE TOPOS (05/12/2017)

§1 The Grothendieck topology

- The general definition of Grothendieck topology

- Examples: (1) The global classical topology; (2) The global Zariski topology; (3) The crystalline topology which we explain now.

- Definition: Let $X$ be a topological space, and let $\mathscr{A}$ be a sheaf of rings on $X$. Let $\mathscr{I} \subseteq \mathscr{A}$ be an ideal of $\mathscr{A}$. A sequence of maps of sets $\gamma_n : \mathscr{I} \to \mathscr{I}$ for $n \geq 0$ is called a PD-structure on $\mathscr{I}$ if for each open $U \subseteq X$ the maps $\gamma_n(U) : \mathscr{I}(U) \to \mathscr{I}(U)$ is a PD-structure on $\mathscr{I}(U)$.

- Fact: Let $X = \text{Spec}(A)$, and let $I \subseteq A$ be an ideal. Denote $\tilde{I}$ the quasi-coherent ideal sheaf associated with $I$. Then to give a PD-structure on $I$ is equivalent to giving a PD-structure on the sheaf $\tilde{I}$. (Key point: PD-structure extends along flat maps, so in particular localizations.)

- Situation: Let $p$ be a prime number, and let $(S, I, \gamma)$, or $(S_0, S, \gamma)$ where $S_0 \subseteq S$ is a closed subscheme with kernel $I$, be a PD-scheme over $\mathbb{Z}_{(p)}$. Let $X \to S_0$ be a map of schemes and suppose that $p$ is nilpotent on $X$.

- The definition of the big and the small crystalline site

§2 The Grothendieck topos

- The definition of a topos
- Examples: (1) The category of sheaves on a topological space, in particular, the category of sets is a topos; (2) The étale topos, the fppf-topos, the fpqc-topos; (3) The crystalline topos which we explain now:
- Proposition: A sheaf on $\mathrm{Cris}(X/S)$ (resp. $\mathrm{CRIS}(X/S)$) is equivalent to the following data: For every morphism $u\colon (U_1, T_1, \delta_1) \to (U, T, \delta)$ we are given a Zariski sheaf $\mathscr{F}_T$ on $T$ and a map $\rho_u\colon u^{-1}(\mathscr{F}_T) \to \mathscr{F}_T$ subject to the following conditions:
  1. If $v\colon (U_2, T_2, \delta_2) \to (U_1, T_1, \delta_1)$ is another map, then $v^{-1}(\rho_u) \circ \rho_v = \rho_{u\circ v}$.
  2. If $u\colon T_1 \to T$ is an open embedding, then $\rho_u^{-1}$ is an isomorphism.

  For a proof see `https://stacks.math.columbia.edu/tag/07IN`.
- Examples: (1) The structure sheaf $\mathscr{O}$ sending $(U, T, \delta) \mapsto \mathscr{O}_T$. (2) The strange sheaf sending $(U, T, \delta) \mapsto \mathscr{O}_U$.

§3 Morphisms between topoi

- A morphism of topoi $f\colon \tilde{X} \to \tilde{Y}$ consists of a pair of adjoint functors
$$(f_*\colon \tilde{X} \to \tilde{Y}, f^*\colon \tilde{Y} \to \tilde{X})$$
  in which $f^*$ commutes with finite inverse limits.
- Definition: A functor $f^{-1}\colon Y \to X$ between two sites is called *continuous* if for any sheaf $\mathscr{F}$ on $X$ the composition $\mathscr{F} \circ f^{-1}$ is a sheaf on $Y$.
- Theorem: Suppose that $f^{-1}\colon Y \to X$ is a continuous functor between two sites, then the functor $\widetilde{f}^{-1}\colon \tilde{X} \to \tilde{Y}$ has a left adjoint $\tilde{f}^*$.
- Definition: A functor $f^{-1}\colon Y \to X$ between two sites is called *cocontinuous* if for any object $U \in Y$ and every covering $\{V_i \to f^{-1}(U)\}$ in $X$, there exists a covering $\{U_j \to U\}$ in $Y$ such that $\{f^{-1}(U_j) \to f^{-1}(U)\}$ refines $\{V_i \to f^{-1}(U)\}$, that is for every $V_i \to f^{-1}(U)$ there exists a $f^{-1}(U_j) \to f^{-1}(U)$ which has a factorization $f^{-1}(U_j) \to V_i$.
- Theorem: Suppose that $f^{-1}\colon Y \to X$ is cocontinuous, then the induced map $\widetilde{f}^{-1}\colon \tilde{Y} \to \tilde{X}$ has a right adjoint $\tilde{f}_*\colon \tilde{Y} \to \tilde{X}$ and $f = (\tilde{f}_*, \widetilde{f}^{-1})$ defines a maps of topoi.
- Theorem: Let $X, Y$ be sites, and let $f^{-1}\colon Y \to X$ be a functor such that
  1. $f^{-1}$ is continuous and cocontinuous.
  2. fibred products and equalizers exist in $Y$ and $f^{-1}$ commutes with those.

  then the induced functor $\tilde{f}^*\colon \tilde{Y} \to \tilde{X}$ commutes with fibred products and equalizers.

- Lemma: The category $\mathrm{CRIS}(X/S)$ has all finite non-empty limits, and the functor

$$\mathrm{CRIS}(X/S) \longrightarrow \mathrm{Sch}_{/X}$$

$$(U, T, \delta) \mapsto U$$

commutes with those.

- Lemma: The category $\mathrm{Cris}(X/S)$ has non-empty limits, and the inclusion

$$i^{-1}\colon \mathrm{Cris}(X/S) \subseteq \mathrm{CRIS}(X/S)$$

commutes with those.

- Corollary: There are morphisms of topoi:

$$(X/S)_{\mathrm{Cris}} \xrightarrow{i} (X/S)_{\mathrm{CRIS}} \xrightarrow{\pi} (X/S)_{\mathrm{Cris}}$$

where $\tilde{i}^* = \tilde{\pi}_* = \widetilde{i^{-1}}$.

- Functoriality: Suppose that we have a PD-morphism $(S, I, \gamma) \to (S', I', \gamma')$ and a diagram:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & S'_0 \end{array}$$

where $S_0 = \mathrm{Spec}\,(\mathcal{O}_S/I)$. Then we have an obvious functor

$$f\colon \mathrm{CRIS}(X/S) \longrightarrow \mathrm{CRIS}(X'/S')$$

This $f$ is both continuous and cocontinuous. This induces a map between topoi

$$(X/S)_{\mathrm{CRIS}} \xrightarrow{f_{\mathrm{CRIS}}} (X'/S')_{\mathrm{CRIS}}$$

Thus we have a map of topoi $f_{\mathrm{Cris}}$ obtained by composition:

$$(X/S)_{\mathrm{Cris}} \xrightarrow{i} (X/S)_{\mathrm{CRIS}} \xrightarrow{f_{\mathrm{CRIS}}} (X'/S')_{\mathrm{CRIS}} \xrightarrow{\pi} (X'/S')_{\mathrm{Cris}}$$

# 8 The Crystalline Topos (12/12/2017)

§1 The global section functor

Let's fix a site $X$. We denote $\hat{X}$ the category of presheaves on $X$ and $\tilde{X}$ the category of sheaves on $X$.

- Let $T$ be an object in $\hat{X}$. Then we define the functor of "taking $T$ sections" to the functor:

$$\Gamma(T,-): \qquad \tilde{X} \longrightarrow \text{(Sets)}$$
$$F \mapsto \mathrm{Hom}_{\tilde{X}}(T,F)$$

If $T$ is taken to be the terminal object $e$ of $\hat{X}$, then we denote $\Gamma(\tilde{X},-)$ or $\Gamma(-)$ for $\Gamma(e,-)$, and this is called the gobal section functor.

- The terminal object in $\hat{X}$ is the sheaf on $X$ which associate with each object in $X$ the singleton, i.e. the set with only one point.

- Examples:

  1. If $X$ is a topological space equipped with the usual topology, then the identity $X \xrightarrow{=} X$ is the terminal object in the category of open embeddings of $X$, so the global section functor associate with a sheaf $\mathscr{F}$ on $Y$ the global section $\mathrm{Hom}_{\hat{X}}(\underline{Y},\mathscr{F})$ which is nothing but the $\mathscr{F}(Y)$ by the by the Yoneda lemma. Moreover, this terminal object certainly does not depend on the choice of $X$.
  2. If $X$ is our site $\mathrm{Cris}(X/S)$, then there is no terminal object in general. Indeed if we take $X$ to be an affine smooth non-empty scheme over $k$, and we take $(S,\mathscr{I},\gamma)$ to be the triple $(\mathrm{Spec}(W_2),(p),\gamma)$, then there is always a deformation $\mathscr{X} \to \mathrm{Spec}(W_2)$ of $X \to \mathrm{Spec}(k)$. Since the ideal $(p)$ is principal, there is a unique PD-structure $\delta$ on $(X,\mathscr{X})$. Since the PD-structure is unique, any automorphism of the pair $(X,\mathscr{X})$ (as a deformation) produces an automorphism of the triple $(X,\mathscr{X},\delta)$. Also any endomorphism of $(X,\mathscr{X})$ as a deformation of induces an isomorphism of $\mathscr{X}$, because the endomorphism is radiciel (univeral homoemorphism), fiberwise étale (indeed fiberwise isomorphism), and flat (because $\mathscr{X} \to \mathrm{Spec}(W_2)$ is flat and all the fibres are flat). Now if $(U,T,\alpha)$ was the terminal object then we have morphism:

  $$(X,\mathscr{X},\delta) \to (U,T,\alpha) \to (X,\mathscr{X},\delta)$$

  in $\mathrm{Cris}(X/W_2)$, where the last arrow is obtained by the smoothness of $\mathscr{X} \to \mathrm{Spec}(W_2)$. Thus we see that in the unique map $(X,\mathscr{X},\delta) \to (U,T,\alpha)$ the map $\mathscr{X} \to T$ is an immersion. Hence the pair $(X,\mathscr{X})$ admits only one automorphism, which is certainly not the case.

- Remark: Let $\tilde{X}$ be a topos induced by a site $X$, and let $e$ be the terminal object. Then the global section functor $\mathscr{F} \mapsto \mathrm{Hom}_{\tilde{X}}(e,\mathscr{F})$ can also be described as follows: It the set of compatible systems $\{\xi_U\}_{U\in X}$, where $\xi_U \in \mathscr{F}(U)$.

- Definition of ringed topos: A ringed topos is a topos plus a ring object in a topos. Let $\hat{X}$ be a topos, let $\mathscr{O}$ be a ring object. Then we write $(\hat{X},\mathscr{O})$ for the ringed topos.

- Let $(\hat{X},\mathscr{O})$ be a ringed topos. Then we denote $O-\mathrm{Mod}$ the category of $\mathscr{O}$ module objects in $\tilde{X}$.

- Examples: (1) When we take $\mathscr{O}$ to be the sheaf which associates to the constant sheaf with value $\mathbb{Z}$, then $\mathscr{O}-$Mod is just the category of abelian sheaves on $X$. (2) For the crystalline topos $(X/S)_{\text{Cris}}$ we take $\mathscr{O}$ to be the sheaf associated with the constant presheaf of value $\mathscr{O}_S(S)$.

- Theorem: For any ringed topos $(\tilde{X}, \mathscr{O})$, the category $\mathscr{O}-$Mod is an abelian category with enough injective objects.

- The global section functor is left exact, so we define the right derived functor to be the crystal cohomology.

- Suppose that we have a commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\quad g \quad} & X \\ \downarrow & & \downarrow \\ (S', \mathscr{I}', \gamma') & \longrightarrow & (S, \mathscr{I}, \gamma) \end{array}$$

  Then there is a map of topoi

$$g_{\text{Cris}}\colon (X'/S')_{\text{Cris}} \to (X/S)_{\text{Cris}}$$

  Moreover the push-forward induces the Grothendieck spectral sequence:

$$E_2^{pq} = H^p((X/S)_{\text{Cris}}, R^q g_* E') \Rightarrow H^{p+q}((X'/S')_{\text{Cris}}, E')$$

  for any $E' \in (X'/S')_{\text{Cris}}$.

- Proposition: There is a natural morphism of topoi

$$u_{X/S}\colon (X/S)_{\text{Cris}} \longrightarrow X_{\text{Zar}}$$

  given by

  1. for any $\mathscr{F} \in (X/S)_{\text{Cris}}$ and $j\colon U \to X$ open embedding we define

$$u_*(\mathscr{F})(U) := \Gamma((U/S)_{\text{Cris}}, \mathscr{F}|_U)$$

  2. for any $E \in X_{\text{Zar}}$ and $(U, T, \delta) \in \text{Cris}(X/S)$ we set

$$(u^*(E)(U, T, \delta) := E(U))$$

- Remark: We can actually see $u_{X/S}$ as a map of ringed topoi if we equip both $(X/S)_{\text{Cris}}$ and $X_{\text{Zar}}$ the constant $\mathscr{O}_S(S)$ ringed topos structure. But we can not equip $X_{\text{Zar}}$ with the $\mathscr{O}_X$-structure, otherwise $u_{X/S}$ would not be a map of ringed topoi.

# 9 THE CRYSTALS AND CALCULUS (19/12/2017)

§1 Crystals

- Definition: Let $\mathscr{C}$ be the site Cris$(X/S)$. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_{X/S}$-modules on $\mathscr{C}$, where $\mathscr{O}_{X/S}$ is the sheaf of rings $(U,T,\delta)\mapsto \mathscr{O}_T$.
  1. We say $\mathscr{F}$ is a crystal if for all map
  $$(U',T',\delta')\xrightarrow{\phi}(U,T,\delta)$$
  in Cris$(X/S)$ the induced map $\phi^*\mathscr{F}_T\to\mathscr{F}_{T'}$ is an isomorphism.
  2. We say that $\mathscr{F}$ is a quasi-coherent crystal if each $\mathscr{F}_T$ is a quasi-coherent $\mathscr{O}_T$-module.
  3. We say that $\mathscr{F}$ is locally free if for each $(U,T,\delta)$ there exists a covering
  $$\{(U_i,T_i,\delta_i)\mapsto(U,T,\delta)\}_{i\in I}$$
  such that $\mathscr{F}|_{(U_i,T_i,\delta_i)}$ is a direct sum of $\mathscr{O}_{X/S}|_{(U_i,T_i,\delta_i)}$.

§2 Sheaves of Differentials

- Definition-Lemma: If $(X_0,X,\delta)$ is a PD-scheme over a scheme $S$ with the structure morphism $f:X\to S$, then there exists an $\mathscr{O}_X$-module $\Omega_{X/S,\delta}$ and a PD-derivation $d\colon\mathscr{O}_X\to\Omega_{X/S,\delta}$ with the property that for any PD-derivation $\varphi\colon\mathscr{O}_X\to M$ there exists a unique $\mathscr{O}_X$-linear map $\Omega_{X/S,\delta}\to M$ which is compatible with $d$ and $\phi$.

- Definition: On Cris$(X/S)$ we have an $\mathscr{O}_{X/S}$-module $\Omega_{X/S}$ whose Zariski sheaf on each object $(U,T,\delta)$, namely the sheaf $(\Omega_{X/S})_T$, is equal to $\Omega_{T/S,\delta}$. Moreover, there is a derivation $d\colon\mathscr{O}_{X/S}\to\Omega_{X/S}$ which is a PD-derivation on each object. This derivation is also universal among all such maps.

- Lemma: Let $(U,T,\delta)$ be an object in Cris$(X/S)$. Let $(U(1),T(1),\delta(1))$ be the product of $(U,T,\delta)$ with itself in Cris$(X/S)$. Let $K\subseteq\mathscr{O}_{T(1)}$ be the ideal corresponding to the closed immersion $T\xrightarrow{\Delta}T(1)$. Then $K\subseteq J(1)$ where $J(1)$ is the ideal of $U(1)\subseteq T(1)$, and we have
$$(\Omega_{X/S})_T=K/K^{[2]}$$

- Lemma: The sheaf of differentials $\Omega_{X/S}$ has the following properties:
  1. $(\Omega_{X/S})_T$ is quasi-coherent,
  2. for any morphism $f\colon(U,T,\delta)\to(U',T',\delta')$ where $T\subseteq T'$ is a closed embedding
  $$f^*(\Omega_{X/S})_{T'}\twoheadrightarrow(\Omega_{X/S})_T$$

§3 Universal Thickening

- Recall: Let $(A, I, \gamma)$ be a PD-triple, let $M$ be an $A$-module, and let $B\colon A \oplus M$ be an $A$-algebra where $M$ is defined to be an ideal of square 0. Let $J := I \oplus M$. Set

$$\delta_n(x+z) := \gamma_n(x) + \gamma_{n-1}(x)z$$

for all $x \in I$ and $z \in M$. Then $\delta$ is a PD-structure on $J$ and

$$(A, I, \gamma) \to (B, J, \delta)$$

is a PD-map.

- Now let $(U, T, \delta) \in \mathrm{Cris}(X/S)$. Set

$$T' := \mathrm{Spec}_{\mathcal{O}_T}(\mathcal{O}_T \oplus \Omega_{T/S,\delta})$$

with $\mathcal{O}_T \oplus \Omega_{T/S,\delta}$ the quasi-coherent $\mathcal{O}_T$-algebra in which $\Omega_{T/S,\delta}$ is a square 0 ideal. Let $J \subseteq \mathcal{O}_T$ be the ideal sheaf of $U \subseteq T$. Set $J' = J \oplus \Omega_{T/S,\delta}$. Then as in the affine case one gets a PD-structure on $J'$ by setting

$$\delta'_n(f, w) = (\delta_n(f), \delta_{n-1}(f)w)$$

Then we get two PD-morphisms: $p_0, p_1 := \mathcal{O}_T \to \mathcal{O}_{T'}$ where

$$p_0(f) = (f, 0)$$

$$p_1(f) = (f, df)$$

or equivalently: $p_0, p_1\colon (U', T', \delta') \to (U, T, \delta)$. There is also a map of PD-schemes

$$i\colon (U, T, \delta) \to (U', T', \delta')$$

which provides a section to both $p_0$ and $p_1$.

§4 Connections

**Definition 3.** A *Connection* on $(X/S)_{\mathrm{Cris}}$ is an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ equipped with an $f^{-1}\mathcal{O}_S$-modules

$$\nabla\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S}$$

such that $\nabla(fs) = f\nabla(s) + s \otimes df$ for all sections $s \in \mathcal{F}$ and $f \in \mathcal{O}_{X/S}$. We can continue defining

$$\nabla\colon \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^n_{X/S} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{n+1}_{\mathcal{O}_{X/S}}$$

by sending $f \otimes m \mapsto \nabla(f) \wedge m + f \otimes dm$. If we write $\nabla(f)$ as $\sum_i f_i \otimes a_i$ with $f_i \in \mathcal{F}$ and $a_i \in \Omega^n_{X/S}$, then the image of $f \otimes m$ can be written as $\sum_i f_i \otimes (a_i \wedge m) + f \otimes dm$. We call the connection integrable if we have $\nabla \circ \nabla = 0$. In this case we have the de Rham complex

$$\mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega^1_{X/S} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega^2_{X/S} \to \cdots$$

**Proposition 9.1.** *Let $\mathscr{F}$ be a crystal in $\mathscr{O}_{X/S}$-modules on* Cris$(X/S)$. *Then $\mathscr{F}$ comes with a canonical Integrable connection.*

*Proof.* We start with $(U,T,\delta) \in \mathrm{Cris}(X/S)$, then we get a thickening $(U',T',\delta')$ with maps

$$(U,T,\delta) \xrightarrow{i} (U',T',\delta') \underset{p_1}{\overset{p_0}{\rightrightarrows}} (U,T,\delta)$$

This provides us isomorphisms:

$$p_0^*\mathscr{F}_T \xrightarrow{c_0} \mathscr{F}_{T'} \xleftarrow{c_1} p_1^*\mathscr{F}_T$$

and the map $c := c_1^{-1} \circ c_0$ is the identity of $\mathscr{F}_T$ via pulling back by $i$. Thus if $s \in \mathscr{F}_T(T)$, then $\nabla(s) := p_1^* s - c(p_0^* s)$ is 0 when pullback via $i$ to $T$. This implies that $\nabla(s) \in \mathrm{Ker}(p_1^*\mathscr{F}_T \to \mathscr{F}_T)$ Thus $\nabla(s) \in \mathscr{F}_T \otimes_{\mathscr{O}_T} \Omega_{T/S}$. The map $\nabla$ is $f^{-1}\mathscr{O}_S$-linear, where $f$ denotes $T \to S$, because all the maps $\mathscr{F}_T \to p_1^*\mathscr{F}$, $\mathscr{F}_T \to p_0^*\mathscr{F}$ and $c$ are all $f^{-1}\mathscr{O}_S$-linear.
For any $f \in \mathscr{O}_T$ we have

$$\begin{aligned}\nabla(fs) &= p_1^*(fs) - cp_0^*(fs)\\ &= (f,df)p_1^* s - (f,0)c(p_0^*(s))\\ &= f\nabla(s) + (0,df)(s\otimes 1)\\ &= f\nabla(s) + s\otimes df\end{aligned}$$

Now let's show that $\nabla$ is integrable.

Step 1. Take $(U,T,\delta) \in \mathrm{Cris}(X/S)$. We define

$$T'' := \mathrm{Spec}_{\mathscr{O}_T}(\mathscr{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta})$$

where the ring structure is defined as

$$(f,w_1,w_2,\eta)(f',w_1',w_2',\eta') = (ff', fw_1' + f'w_1, fw_2' + f'w_2, f\eta' + f'\eta + w_1 \bigwedge w_2' + w_1' \bigwedge w_2)$$

Let

$$J'' := J \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega^2_{T/S,\delta}$$

We can define a PD-structure on $J''$ by setting

$$\delta''(f,w_1,w_2,\eta) = (\delta_n(f), \delta_{n-1}(f)w_1, \delta_{n-1}(f)w_2, \delta_{n-1}(f)\eta + \delta_{n-2}(f)w_1 \bigwedge w_2)$$

There are 3 maps $q_0, q_1, q_2$ of PD-triples

$$(U'',T'',\delta'') \to (U,T,\delta)$$

defined by

$$q_0(f) := (f,0,0,0)$$

$$q_1(f) := (f,df,0,0)$$

$$q_2(f) := (f,df,df,0)$$

There are also three projections $\mathcal{O}_{T'} \to \mathcal{O}_{T''}$ defined by

$$q_{01}(f,w) = (f,w,0,0)$$

$$q_{12}(f,w) = (f,df,w,dw)$$

$$q_{02}(f,w) = (f,w,w,0)$$

These are also PD-maps. Moreover we have the following relations.

$$q_0 = q_{01} \circ p_0 \qquad q_1 = q_{01} \circ p_1$$

$$q_1 = q_{12} \circ p_0 \qquad q_2 = q_{12} \circ p_1$$

$$q_0 = q_{02} \circ p_0 \qquad q_2 = q_{02} \circ p_1$$

Step 2. Take $\mathcal{F}$ a crystal on $\mathrm{Cris}(X/S)$. Then there is a commutative diagram:

$$\begin{array}{ccc} q_0^*\mathcal{F}_T & \xrightarrow{q_{01}^* c} & q_1^*\mathcal{F}_T \\ & {\scriptstyle q_{02}^* c}\searrow \quad \swarrow{\scriptstyle q_{12}^* c} & \\ & q_2^*\mathcal{F}_T & \end{array}$$

whose commutativity comes from the commutativity of the following small diagrams:

$$\begin{array}{ccccc} q_0^*\mathcal{F}_T & \longrightarrow & q_{01}^*\mathcal{F}_{T'} & \longleftarrow & q_1^*\mathcal{F} \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ q_{02}^*\mathcal{F}_{T'} & \longrightarrow & \mathcal{F}_{T''} & \longleftarrow & q_{12}^*\mathcal{F}_T \\ & \nwarrow & \uparrow & \nearrow & \\ & & q_2^*\mathcal{F}_T & & \end{array}$$

Step 3. For $s \in \Gamma(T, \mathcal{F}_T)$ we have $c(p_0^* s) = p_1^* s - \nabla(s)$. Write $\nabla(s) = \sum_i p_1^* s_i \cdot w_i$ where $s_i \in \mathcal{F}_T$ and $w_i \in \mathcal{O}_{T'}$. Then we have

$$\begin{aligned}(q_{12}^* c) \circ (q_{01}^* c)(q_0^* s) &= (q_{12}^* c) \circ (q_{01}^* c)(q_{01}^*(p_0^* s)) \\ &= (q_{12}^* c)(q_{01}^*(p_1^* s - \sum_i p_1^* s_i \cdot w_i)) \\ &= (q_{12}^* c)(q_{12}^*(p_0^* s) - \sum_i q_{12}^*(p_0(s_i)) q_{01}(w_i)) \\ &= q_{12}^*(p_1^* s - \sum p_1^* s_i \cdot w_i) - \sum_i q_{12}^*(p_1^* s_i - \nabla(s_i)) q_{01}(w_i) \\ &= (q_2^* s - \sum_i q_2^* s_i \cdot q_{12}(w_i)) - \sum_i q_2^* s_i \cdot q_{01}(w_i) + \sum_i q_{12}^*(\nabla(s_i)) \cdot q_{01}(w_i)\end{aligned} \tag{9.1}$$

On the other hand one has

$$q_{02}^* c(q_0^* s) = q_2^* s - \sum_i q_2^* s_i \cdot q_{02}(w_i) \tag{9.2}$$

Clearly we have $q_{01}(w_i)+q_{12}(w_i)-q_{02}(w_i)=dw_i$. Thus taking (9.2)-(9.1) we get

$$\sum_i q_2^* s_i \cdot dw_i - \sum_i q_{12}^*(\nabla(s_i))\cdot q_{01}(w_i)$$

If one looks into the formula, it is precisely $\nabla\circ\nabla(s)$. □

# 10 The Equivalence between Crystals and Connections (16/01/2018)

- Situation: Let $p$ be a prime number, and let $(A,I,\gamma)$ be a PD-triple in which $A$ is a $\mathbb{Z}_{(p)}$-algebra. Let $A\to C$ be a ring map such that $IC=0$ and such that $p$ is nilpotent in $C$. We write $X=\operatorname{Spec}(C)$ and $S=\operatorname{Spec}(A)$. Choose a polynomial ring $P=A[x_i]$ over $A$ and a surjection $P\twoheadrightarrow C$ of $A$-algebras with kernel $J:=\operatorname{Ker}(P\twoheadrightarrow C)$. Set

  $$D:=\varprojlim_e D_{P,\gamma}(J)/p^e D_{P,\gamma}(J)$$

  for the $p$-adically completed divided envelop. This ring $D$ comes with a triple $(D,\bar{J},\bar{\gamma})$. We have seen in the exercise that $(D/p^eD,\bar{J}/\bar{J}\cap p^eD,\bar{\gamma})$ is the PD-envelop of $(P/p^eP,J/p^eJ)$ for $e$ large. On the other hand, we have

  $$\Omega_D=\varprojlim_e \Omega_{D_e/A,\bar{\gamma}}=\varprojlim_e \Omega_{D/A,\bar{\gamma}}/p^e\Omega_{D/A,\bar{\gamma}}$$

  On the other hand we have

  $$\Omega_{D/A,\bar{\gamma}}=\Omega_{P/A}\otimes_P D$$

  as we have seen before. So $\Omega_{D/A,\bar{\gamma}}$ is a free $D$-module on the basis $\{dx_i\}_{i\in I}$, and any element in $\Omega_D$ can be written uniquely as a sum (possibly infinite) of the form $\sum_{i\in I}a_i dx_i$.

- Definition: Let

  $$D(n):=\varprojlim_e D_{P\otimes_A\cdots\otimes_A P,\gamma}(J(n))/p^e D_{P\otimes_A\cdots\otimes_A P,\gamma}(J(n))$$

  where $J(n)$ is the kernel of $P\otimes_A P\otimes_A\cdots\otimes_A P\twoheadrightarrow C$. We set

  $$\bar{J}(n):=\text{the divide power ideal of } D(n)$$

  $$D(n)_e:=D(n)/p^eD(n)$$

  $$\Omega_{D(n)}:=\varprojlim_e \Omega_{D(n)_e/A,\bar{\gamma}}=\varprojlim_e \Omega_{D(n)/A,\bar{\gamma}}/p^e\Omega_{D(n)/A,\bar{\gamma}}$$

- Quasi-nilpotent connections:
  Definition: We call a pair $(M,\nabla)$ a quasi-nilpotent connection of $D/A$ if $M$ is a $p$-adically complete $D$-module, $\nabla$ is an integrable connection

  $$\nabla\colon M\longrightarrow M\widehat{\otimes}_D\Omega_D$$

  and *topologically quasi-nilpotent*, that is, if we write $\nabla(M)=\sum\theta_i(m)dx_i$ for some operators $\theta_i\colon M\to M$, then we have that for any $m\in M$ there are only finitely many pairs $(i,k)$ such that $\theta_i^k(m)\notin pM$.

- Theorem: There is an equivalence:

$$\boxed{\text{quasi-coherent crystals on } \mathrm{Cris}(X/S)} \Longleftrightarrow \boxed{\text{quasi-nilpotent connections of } D/A}$$

- Proof: We will construct two functors in two opposite directions and then claim without proof that that they are inverse to each other.

  The functor from the left to the right: Given a quasi-coherent crystal $\mathscr{F}$ on $\mathrm{Cris}(X/S)$, we consider the sequence of objects $(X, T_e, \delta_e)$ where $T_e := \mathrm{Spec}(D_e)$. If we take value of $\mathscr{F}$ on each $T_e$, then we get a sequence of $D$-modules $M_e$ satisfying that

$$M_e = M_{e+1} \otimes_{\mathbb{Z}/p^{e+2}\mathbb{Z}} \mathbb{Z}/p^{e+1}\mathbb{Z}$$

  Let $M := \varprojlim_e M_e$ then $M$ is a $p$-adically complete module.
  By 9.1 there is a canonical connection on

$$\nabla\colon \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_{X/S}} \Omega^1_{X/S}$$

  By taking values on each $T_e$ and then taking limit, we get an integrable connection

$$\nabla\colon M \to M \widehat{\otimes}_D \Omega^1_D$$

  We have to show that this connection is topologically quasi-nilpotent. We do the same procedure for $D(n)$ and get a $p$ -adically complete $D(n)$-module $M(n)$. Since $\mathscr{F}$ is a crystal, we have isomorphisms:

$$M \widehat{\otimes}_{D,p_0} D(1) \longrightarrow M(1) \longleftarrow M \widehat{\otimes}_{D,p_1} D(1)$$

  Let $c$ denote the arrow which goes directly from the left to the right. Pick $m \in M$. Write $\xi_i := x_i \otimes 1 - 1 \otimes x_i$. Then we have a unique expression of $c(m \otimes 1)$ in terms of $\xi_i$:

$$c(m \otimes 1) = \sum_K \theta_K(m) \otimes \prod \xi_i^{[k_i]}$$

  where $K$ runs over all multi-indices $K = (k_i)$ with $k_i \geq 0$ and $\sum k_i < \infty$. This is due to the following
  **Lemma:** The projection

$$P \to P \otimes_A \cdots \otimes_A P$$
$$f \mapsto f \otimes 1 \cdots \otimes 1$$

  induces an isomorphism:

$$D(n) = \varprojlim_e D\langle \xi_i(j) \rangle / p^e D\langle \xi_i(j) \rangle$$

  where $\xi_i(j) := x_i \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1$.
  **Proof of the Lemma:** Indeed we have

$$P \otimes_A \cdots \otimes_A P = P[\xi_i(j)]$$

where $\xi_i(j)$ are considered as indeterminates, and $J(n)$ is generated by $J$ and those $\xi_i(j)$. Then we apply the last item of §3. **End of the Proof**

Set $\theta_i = \theta_K$ where $K$ has 1 in the $i$-th spot and 0 elsewhere. Recall the construction of the canonical connection on the crystal $\mathscr{F}$. For each thickening like $(X, T_e, \delta_e)$ we construct a thickening $(X, T'_e, \delta'_e)$ with two projections: $p, q\colon T'_e \to T_e$. As $\mathscr{F}$ is a crystal, there are isomorphisms:

$$p^*\mathscr{F}_{T_e} \xrightarrow{c_0} \mathscr{F}_{T'_e} \xleftarrow{c_1} q^*\mathscr{F}_{T_e}$$

We wrote $c$ for the map which goes directly from the left to the right. For any section $s \in \mathscr{F}_{T_e}$ we defined

$$\nabla(s) := q^*s - c(p^*s)$$

We have a unique map $\phi\colon T'_e \to \mathrm{Spec}\,(D(1)_e)$ whose compositions with the two canonical projections of $D(n)_e$ are the two projections $p$ and $q$. Indeed this follows from the following

**Lemma:** We have

$$D(n) = \coprod_{j=0,\cdots,n} D$$

$$D(n)_e = \coprod_{j=0,\cdots,n} D_e$$

in $\widehat{\mathrm{Cris}}(C/A)$, where $e$ is supposed to be sufficiently large.

**Proof of the Lemma:** If $(B \twoheadrightarrow C, \delta) \in \widehat{\mathrm{Cris}}(X/S)$, then we have

$$\begin{aligned}
&\mathrm{Hom}_{\widehat{\mathrm{Cris}}(X/S)}(D(n)_e, B)\\
=&\{f \in \mathrm{Hom}_A((P_e \otimes_A \cdots \otimes_A P_e, J(n)), (B, \mathrm{Ker}(B \twoheadrightarrow C))) | f \text{ induces identity on } C\}\\
=&\prod_n \{f \in \mathrm{Hom}_A((P_e, J), (B, \mathrm{Ker}(B \twoheadrightarrow C))) | f \text{ induces identity on } C\}\\
=&\prod_n \mathrm{Hom}_{\widehat{\mathrm{Cris}}(X/S)}(D_e, B)
\end{aligned}$$

and we have the same equation for $D(n)$. **End of the Proof**

Thus

$$\begin{aligned}
\nabla(m) =& \phi^*(p_1^*(m) - c(p_0^*(m)))\\
=& \phi^*(m \otimes 1 - c(m \otimes 1))\\
=& \phi^*(m \otimes 1 - m \otimes 1 - \sum_i \theta_i(m)\xi_i)\\
=& \sum_i \theta_i(m) dx_i
\end{aligned}$$

As in 9.1 we have the equality:

$$q_{02}^* c = q_{12}^* c \circ q_{01}^* c$$

Applying it to $m \otimes 1$ we get

$$\sum_{K"} \theta_{K"}(m) \otimes \prod \zeta_i"^{[k_i"]} = \sum_{K,K'} \theta_{K'}(\theta_K(m)) \otimes \prod \zeta_i'^{[k'_i]} \prod \zeta_i^{[k_i]}$$

in $M\widehat{\otimes}_{D,q_2}D(2)$, where

$$\zeta_i = x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1$$

$$\zeta_i' = 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes x_i$$

$$\zeta_i" = x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i$$

We have $\zeta_i" = \zeta_i + \zeta_i'$ and that

$$D(2) = q_2(D)\langle \zeta_i, \zeta_i' \rangle$$

Comparing the coefficients we get

1. $\theta_i \circ \theta_j = \theta_j \circ \theta_i$
2. $\theta_K(m) = (\prod \theta_i^{k_i})(m)$

If we mod $p$, then there could only be finitely many $\theta_K(m)$ survive. Thus there are only finitely many $\theta_i^k(m)$ which do not line in $pM$.

## 11 Crystals and HPD-Stratifications (23/01/2018)

- Finish the proof the equivalence between quasi-coherent crystals and quasi-nilpotent connections in quasi-coherent modules.

- Definition: The conventions and notations are as in the last lecture. Suppose that we have a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow f & & \downarrow \\ S_0 & \longrightarrow & S \end{array}$$

  Set $D$, $D_{Y,\gamma}(J)$ as before. A quasi-coherent HPD-stratification associated with this diagram is a $p$-adically complete quasi-coherent $\mathcal{O}_D$-module $M$ equipped with an isomorphism

$$\phi\colon p_0^* M \xrightarrow{\cong} p_1^* M$$

  satisfying the cocycle condition:

$$p_{01}^*\phi \circ p_{12}^*\phi = p_{02}^*\phi$$

  where $p_0, p_1$ are the two projections $D(1)$ to $D$ and $p_{01}, p_{12}, p_{02}$ are the three projections from $D(2)$ to $D(1)$, where the pullbacks are taking by the completed tensor product.

- Theorem: Assumptions and conventions being as above, assume further that $f$ is smooth, then there is an equivalence of categories between the category of quasi-coherent crystals on Cris($X/S$) and the category of HPD-Stratifications with respect to a diagram as as above.

## 12 The Comparison Theorem (I) (30/01/2018)

- A brief introduction to spectral sequences. Firstly, make the definition of a spectral sequence. Then construct the spectral sequence associated with a complex with a descending filtration. Finally, introduce the two spectral sequences coming from a double complex.
- A brief Introduction to simplicial objects and cosimplicial objects.
- Introduce the Dold-Kan theorem. Explain how one gets a cochain complex out of a cosimplicial object.
- Assuming two key lemma:
  1. The $p$-adic poincaré lemma;
  2. For any quasi-coherent crystal $\mathscr{F}$, the Čech complex associated with the Čech covering $D \twoheadrightarrow S$ is quasi isomorphic to $R\Gamma(\mathrm{Cris}(X/S), \mathscr{F})$. Note that since $D(n)$ is actually the $n$-th product of $D$ in $\widehat{\mathrm{Cris}}(X/S)$, the complex is of the form

  $$\mathscr{F}(D) \to \mathscr{F}(D(1)) \to \mathscr{F}(D(2)) \cdots$$

  One can prove the comparison theorem using the spectral sequence associated with the following double complex:

  $$M \hat{\otimes}_D \Omega^p_{D(q)}$$

## 13 The Comparison Theorem (II) (02/02/2018)

1. Finish the proof of the two main lemmas.
2. Introduce the comparison theorem in the non-affine case. Note that in this case, one can only do it assuming that $S$ is killed by a power of $p$. This sucks!!!